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Unimodular ϵ-Hermitian Forms Revisited

EVA BAYER–FLUCKIGER*

Department of Mathematics,
Université de Genève, 2–4, rue du Lièvre, 1211 Geneva 24, Switzerland

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Let K be an algebraic number field with a nontrivial involution, and let S be a Dedekind set of primes of K (cf. [9, Sect. 21]) which is invariant under this involution. Let A be the set of S-integers of K. We shall study the classification, up to isometry, of unimodular ε-hermitian forms (where $\varepsilon = \pm 1$) on projective A-modules of finite rank. This problem has been considered in [1] in the special case where A is the ring of integers of K (i.e., S contains every finite prime of K). However, the more general situation naturally arises in knot theoretical problems (see, e.g., Levine [8]).

In the present note we shall see that not only the results of [1] generalize to S-integers, but in fact the case where some finite prime of K does not belong to S is much simpler. Every form of rank greater than two behaves like an indefinite form (even if all the signatures are maximal) and a complete set of invariants is given by rank, signatures, determinant (which is a rank one form), and, in the skew-hermitian case, a finite set of pfaffians. The proof of this uses a generalization, due to Kneser, of the strong approximation theorem of G. Shimura, and also some results of Wall.

1

Let F be the fixed field of the involution, and let S_0 be the set of primes p of F such that $p = P \cap F$ for some P in S. Assume that S_0 contains almost all finite primes of F. Let Ω_0 denote the set of all primes, finite and infinite, of F. Let us denote F_p the completion of F at p, B_p the ring of integers of F_p, $K_p = K \otimes_F F_p$ and $A_p = AB_p$.

Let (V, h) be a nonsingular hermitian or skew-hermitian form. We shall say that S_0 (or S) is an *indefinite set of primes* for (V, h) if there exists at least one prime p in $\Omega_0 \backslash S_0$ such that $(V, h)_p = (V, h) \otimes_K K_p$ is isotropic (i.e.,

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there exists a nonzero x in V_p such that $h(x,x)=0$). A *lattice* L in (V,h) is a finitely generated projective A-module such that $L\otimes_A K=V$ and that the restriction of h to L takes values in A.

The following is a consequence of a result of Kneser (cf. [5, Satz 2; 6]).

THEOREM 1. *Let* (V,h) *be a nonsingular hermitian or skew-hermitian form. Assume that* $\dim(V)>1$ *and that* S *is an indefinite set of primes for* (V,h). *Then an* SU*-genus of* A*-lattices consists of only one* SU*-class.*

This generalizes Shimura's theorem [10, 5.19]. One can also use Shimura's proof, but instead of applying Eichler's theorem [2, Satz 5] one has to apply a generalization of this theorem (cf. [11, Proposition 5.8]).

2

Let (L,h) be a lattice. The *determinant* of (L,h) is the rank one form $\det(L,h):\Lambda^n L\times\Lambda^n L\to A$, $\det(L,h)(x_1\wedge\cdots\wedge x_n, y_1\wedge\cdots\wedge y_n)=\det(h(x_i,y_j)_{ij})$, where $n=\operatorname{rank}_A(L)$.

We shall say that (L,h) is *unimodular* if the adjoint of h, $\operatorname{ad}(h)$: $L\to\operatorname{Hom}_A(L,A)$, given by $\operatorname{ad}(h)(x)(y)=h(y,x)$, is bijective.

Assume that there exists $\alpha\in A$ such that $\alpha+\bar{\alpha}=1$ (this hypothesis is satisfied in the knot theoretical applications). This implies that no dyadic prime of F ramifies in K (cf. [1, Remark 3.13]).

Let (L,h) be a unimodular, skew-hermitian lattice of even rank and let p be a prime of F which ramifies in K, $P^2=pA$. Then the involution on A/P is trivial (cf. [4, Sect. 5]). The skew-hermitian form h induces a nonsingular skew-symmetric form $\bar{h}$ on $\bar{L}=L/PL$. Let us denote $\mathrm{P}f_p(L,h)$ a *pfaffian* of this form. If (M,h) is another lattice, and if $\varphi:\bar{L}\to\bar{M}$ is an isometry, then $\mathrm{Pf}_p(M,h)=\det(\varphi)\,\mathrm{Pf}_p(L,h)$.

The unimodular lattices for which S is an indefinite set of primes are classified by rank, signatures, determinant, and pfaffians.

THEOREM 2. *Assume that* S *is an indefinite set of primes for the unimodular,* ε*-hermitian lattices* L *and* M. *Then* L *and* M *are isometric if and only if one of the following holds:*

(a) $\varepsilon=+1$, L, *and* M *have same rank, signatures, and isometric determinants.*

(b) $\varepsilon=-1$, L, *and* M *have same rank, signatures, and there exists an isometry* f *between* $\det(L)$ *and* $\det(M)$ *such that* $\det(f)\ \mathrm{Pf}_p(L)\equiv\mathrm{Pf}_p(M)$ $\mathrm{mod}\ P$ *for all primes* p *of* F *such that* $pA\equiv P^2$.

This is a generalization of [1, Corollary 4.10].

Remark 1. If S does not contain all finite primes of K and if $\dim V > 2$ (where $V = L \otimes_A K$), then the hypothesis of the theorem is always satisfied. Indeed, $(V, h)_p$ is isotropic when p is a finite prime and $\dim V > 2$. Moreover, if p is split (i.e., $pA = P\bar{P}$, where $P \neq \bar{P}$), then $(V, h)_p$ is isotropic also for $\dim(V) = 2$ (notice that $K_p = F_p \times F_p$ if p is split). Therefore the theorem holds in all dimensions provided $\Omega_0 \backslash S_0$ contains a split finite prime.

The following lemma can be deduced from Wall's results (cf. [12]).

LEMMA. *Let* $x \in A_p$, $x\bar{x} = 1$. *Then there exists a* $\psi \in U(V, h)_p$ *such that* $\det \psi = x$, $\psi(L_p) = L_p$, *if and only if either* $\varepsilon = 1$, *or* $\varepsilon = -1$, *and* $x \equiv 1 \bmod P$, *where* $pA = P^2$.

Sketch of Proof. If $\varepsilon = 1$ or $\varepsilon = -1$ and p is unramified, it is easy to obtain the lemma from the classification of unimodular forms over A_p (cf. [10, 4.18; 12, pp. 431–433]). Let us assume that $\varepsilon = -1$ and that p is ramified. Then $(L, h)_p$ is hyperbolic (cf. [12, p. 234; 4, Proposition 8.1.b]). Set $\bar{L} = L/PL$. Then $\bar{L}$ supports a nonsingular skew-symmetric form $\bar{h}$. Let $\psi \in U(V, h)_p$ such that $\psi(L_p) = L_p$. Then ψ induces an automorphism of $(\bar{L}, \bar{h})$, the determinant of which must be $+1$, therefore $x = \det(\psi) \equiv 1 \bmod P$. Conversely, if $x \equiv 1 \bmod P$, then $x = y^2$ with $y \in A_p$ by Hensel's lemma (p is nondyadic) and $y \equiv \pm 1 \bmod P$. This, together with $x\bar{x} = 1$ implies $y\bar{y} = 1$. Let $e, f \in L$ be the basis of a hyperbolic plane $H \subset L$. Let us define $\psi(e) = ye$, $\psi(f) = yf$, and let ψ be the identity on the orthogonal complement of H. Then $\psi \in U(V, h)_p$, $\psi(L_p) = L_p$ and $\det(\psi) = x$.

Proof of Theorem 2. The conditions of the theorem are clearly necessary. Let us prove that they are also sufficient. By Landherr's theorem (cf. [7]) we can assume that L and M are both lattices in (V, h). By [12, Proposition 6] this implies that L and M are in the same genus. We shall now use a similar argument to Shimura's proof of [10, Proposition 5.27]. Let $f: \det(L) \to \det(M)$ be an isometry and let $a = \det(f)$. Then $a\bar{a} = 1$. There exists an element ψ of $U(V, h)$ such that $\det \psi = a$.

Let $N = \psi(L)$. For every prime p of F there exists an element φ_p of $U(V, h)_p$ such that $\varphi_p(M_p) = N_p$. On the other hand the existence of f implies that we have an element F of $GL(V)$ such that $F(L) = M$ and $\det(F) = a$. This implies that $\det(\varphi_p)$ is a unit for all p. We also have $\mathrm{Pf}_p(N) = \mathrm{Pf}_p(M)$ if p is ramified, therefore $\det(\varphi_p) \equiv 1 \bmod P$, where $P^2 = pA$. By the lemma this implies that there exists an element ϕ_p of $U(V, h)_p$ such that $\phi_p(M_p) = M_p$ and that $\det(\phi_p) = \det(\varphi_p)^{-1}$. Therefore N and M are in the same SU-genus, so they are SU-equivalent by Theorem 1.

Remark 2. It is easy to check that the other results of [1, Section 4] can also be generalized to the case of S-integers. So we have class number formulas (Proposition 4.8, of course one has to replace C_k by the group of classes of A-ideals), decomposition theorems into lattices of rank at most 2 if $\varepsilon = \pm 1$ (Proposition 4.11) and at most 4 if $\varepsilon = -1$ (Proposition 4.12) and cancellation (Proposition 4.13).

Notice that L. Gerstein's decomposition theorem, for non necessarily unimodular lattices, also generalizes to S-integers: if S does not contain all finite primes of K, then every hermitian A-lattice is isometric to the orthogonal sum of lattices of rank at most 4. The proof is as in [3, Theorem 3.14], except that one has to apply Theorem 1 instead of Shimura's theorem.

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